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A New Concept in Mechanics Based on the Notions of Space, Time, and Energy

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Abstract—A new concept in mechanics is proposed based on the notions of space, time, and energy. The energy is represented by the sum of thirteen terms expressed as products of invariants of the Lagrangian equations of motion and scalar factors corresponding to the physical properties of materials. Differential equations of motion and equilibrium are derived from the frame indifference condition for energy, in which the Eulerian variables are the sought functions and the Lagrangian variables are the arguments. Assumptions are made under which these equations can be transformed into the Poisson and Laplace equations. The law of conservation of energy is used to obtain dependences between the Lagrangian stresses and strains in the reversible deformation region; they are compared with the expressions used in the theory of elasticity. It is discussed whether the point of reference for the mean stresses may be chosen with regard to the volume energy density of particles in their initial state, as well as whether elastic deformation can be described using a single constant. As an example, the energy model and the equations of motion in the Lagrangian form are applied to describe the transition from reversible to irreversible deformation. The developed model differs from the classical one by using two independent infinitesimal operators for the time and space. It is shown that Newton's law of inertia can be regarded as a variant for the determination of the generalized forces characterizing the kinetic energy change in a body with increments in the distance between the origin of the observer coordinate system and the center of mass of the body. The application of the Lagrangian variables and the superposition principle for describing any spatial motions (including for absolutely rigid bodies), which can be used in the dynamic analysis of linkages and other mechanisms, is validated. The multiple choice of generalized forces for absolutely rigid bodies, including when passive forces arise, is considered. The method of dynamic analysis of mechanisms developed on the basis of the energy model allows the law of conservation of energy to be satisfied for any part of the studied system in an arbitrary time interval.

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1. MATHEMATICAL DEFINITION OF THE NOTION OF ENERGY

The notion of space allows one to describe the position of objects in it by an ordered set of numbers, in particular $x_i \in (x, y, z)$ in the Cartesian coordinate system. Using the time t, it is possible to describe the change in the position of a particle in this space in the form of the equations

$$x_i = x_i(\alpha_p, t), \tag{1}$$

where $\alpha_p \in (\alpha, \beta, \gamma)$ are the Lagrangian coordinates [1], for which we will use the initial particle coordinates $\alpha = x|_{t=0}$, $\beta = y|_{t=0}$, $\gamma = z|_{t=0}$.

Any external forces should be accounted for in equations of motion (1); they carry all information about changes in each particle of the observed mechanical system. According to Aristotle, energy is a generalized scalar characteristic of any kind of motion and therefore must take into account all invariant characteristics of system (1). In the most general case, it has thirteen invariants ξ_i ($i = 1, ..., 13$), including three magnitudes of the displacement, velocity and acceleration vectors:

$$\xi_1 = |u| = \sqrt{u_x^2 + u_y^2 + u_z^2},$$
$$\xi_2 = |v| = \sqrt{v_x^2 + v_y^2 + v_z^2}, \tag{2}$$
$$\xi_3 = |w| = \sqrt{w_x^2 + w_y^2 + w_z^2},$$

three invariants of the tensor $\partial x_i / \partial \alpha_p \equiv x_{i,\alpha_p}$, that define particle deformation (dimensionless generalized coordinates):

$$\xi_5 = x_\alpha + y_\beta + z_\gamma,$$
$$\xi_6 = x_\alpha^2 + x_\beta^2 + x_\gamma^2 + y_\alpha^2 + y_\beta^2 + y_\gamma^2 + z_\alpha^2 + z_\beta^2 + z_\gamma^2, \quad (3)$$
$$\xi_7 = |x_{i,p}| = \delta V/\delta V_0,$$

and three invariants of the strain rate tensor $\partial x_{i,t}/\partial\alpha_p \equiv x_{i,t\alpha_p}$ (velocities of the dimensionless generalized coordinates):

$$\xi_8 = x_{t\alpha} + y_{t\beta} + z_{t\gamma},$$
$$\xi_9 = x_{t\alpha}^2 + x_{t\beta}^2 + x_{t\gamma}^2 + y_{t\alpha}^2 + y_{t\beta}^2 + y_{t\gamma}^2 + z_{t\alpha}^2 + z_{t\beta}^2 + z_{t\gamma}^2, \quad (4)$$
$$\xi_{10} = \left|x_{i,tp}\right|.$$

Other invariants are the path

$$s = \int_{t_1}^{t_2} v\,\mathrm{d}t \quad (5)$$

and three time integrals of the three strain rate tensor invariants (4)

$$\xi_{11} = \int \xi_8 \mathrm{d}t, \ \xi_{12} = \int \xi_9^{1/2}\mathrm{d}t, \ \xi_{13} = \int \xi_{10}^{1/3}\mathrm{d}t. \quad (6)$$

The latter include the accumulated strain (Odqvist parameter) that characterizes hardening and the degree of plasticity exhaustion in the material [2].

The second time derivatives of x_{i,α_p} are not considered, because according to the fundamental postulate of mechanics [1] the behavior of particles is determined only by their position and velocities. The invariants ξ_i ($i = 1, ..., 13$) are basic for the mathematical definition of the notion of energy.

Using the notion of energy and the law of conservation of energy, we can determine necessary properties for predicting the behavior of a system subjected to externally applied forces and develop methods for determining these properties by actually observed motions.

In view of the different nature of the coordinates $\alpha_p \in (\alpha, \beta, \gamma)$ and time t, two independent operators must be used to denote infinitesimal increments of any function $f(\alpha, \beta, \gamma, t)$: the operator d is for the time increments:

$$\mathrm{d}f(\alpha, \beta, \gamma, t) = f_t \mathrm{d}t,$$

and the operator δ is for the space increments:

$$\delta f(\alpha, \beta, \gamma, t) = f_\alpha \delta\alpha + f_\beta \delta\beta + f_\gamma \delta\gamma,$$

where $\delta\alpha, \delta\beta, \delta\gamma$ are the infinitesimal increments of the Lagrangian variables.

Energy carriers are particles, including infinitesimal particles of volume $\delta V_0 = \delta\alpha\delta\beta\delta\gamma$. We will use the notation $\delta E = \delta E(\xi_i)$, for a generalized scalar function, where the operator δ emphasizes the local character of the function. Modern ideas are consistent with the hypothesis that the function $\delta E(\xi_i)$ can be represented as a sum of 13 terms depending only on one of the invariants

$$\delta E = \sum_{i=1}^{13} \delta E_i(\xi_i), \quad (7)$$

where each term can be represented as the product of the corresponding invariant by the volume δV_0 and the scalar factor k_i, that ensures the equality of the dimensions of the terms:

$$\delta E = \sum_{i=1}^{13} \delta E_i(\xi_i) = \sum_{i=1}^{13} k_i \xi_i \delta V_0 . \quad (8)$$

Based on the kinematic invariants, we can postulate the existence of 13 different types of energy that generally characterize the motion of absolutely rigid or deformable bodies and reflect the effect of various kinematic factors (which cause changes in the position, volume, and shape of the constituent particles) on energy.

2. ENERGY-BASED DEFINITION OF FORCE

For the practical application of the law of conservation of energy, it is more convenient to determine not the energy itself, but the increments of its components $\mathrm{d}\delta E_i(\xi_i)$, which depend on the invariants ξ_i ($i = 1, ..., 13$) that are functions of the corresponding vector or tensor components. The set of the kinematic parameters used to find the invariants $\xi_i = \xi_i(q_j)$ is formed by the generalized coordinates q_j. Then for the time increment $\mathrm{d}\delta E_i(k_i, \xi_i(q_j))$ we should write the following expression with the increments chosen to describe the motion of the coordinates q_j:

$$\mathrm{d}\delta E_i(k_i, \xi_i(q_j)) = \frac{\partial \delta E_i}{\partial \xi_i}\frac{\partial \xi_i}{\partial q_j}\mathrm{d}q_j = \delta Q_{ij}\mathrm{d}q_j. \quad (9)$$

The right-hand side of Eq. (9) coincides in form with the definition of generalized forces P_j as a measure of power increment with increments of the generalized kinematic coordinates q_j [1, 2] $\mathrm{d}A = \sum P_j \mathrm{d}q_j$. Therefore, Eq. (9) can be regarded as an extension of the notion of generalized forces to various types of energy E_i. The multiplier δQ_{ij} in Eq. (9) is the energy-based definition of the generalized local force

$$\delta Q_{ij} = \frac{\partial \delta E_i}{\partial q_j} = k_i \frac{\partial \xi_i}{\partial q_j}\delta V_0, \quad (10)$$

which characterizes the rate of change of particular energy $\delta E_i(\xi_i(q_j))$ of a particle with changing kinematic coordinate q_j.

The choice of generalized coordinates is generally subjective. Depending on the problem solved, each of the invariants ξ_i can be represented in terms of vector projections in the accepted coordinate system of the observer (Cartesian, cylindrical, spherical, etc.), which is naturally subjective. But since the product of generaliz-

ed forces and corresponding kinematic coordinates must be a scalar quantity, the generalized forces retain the mathematical properties of the corresponding kinematic coordinates. In special cases, the force Q_{ij} can be a vector if the parameters q_j are vector projections (linear or angular displacements), a stress tensor if the kinematic coordinates are strain tensor components $\partial x_i/\partial\alpha_p \equiv x_{i,\alpha_p}$, or a scalar if a scalar, e.g., path s or the square of velocity v^2, is chosen as the kinematic parameter. Scalar coefficients $k_1 - k_{13}$ can also be considered as the forces characterizing the change in the volume density of particular types of energy E_i with increments of independent invariants $\xi_1 - \xi_{13}$:

$$k_i = \frac{\partial E_i}{\partial \xi_i}\frac{1}{\delta V_0}.$$

Only by using the product of the components of generalized forces Q_{ij} (10) and the corresponding increments of generalized coordinates $\mathrm{d}q_j$ (or velocities $q_{j,t}$), in accordance with Eq. (9), we can obtain the scalar characteristic of the state (or process), i.e., the energy increment (or the rate of energy change). However, to do this, we must know the complete set of generalized forces and corresponding generalized kinematic coordinates. The determination of the generalized forces through energy equation (9) defines their physical meaning and the main purpose, which is the determination of increments or the rate of change of different types of energy.

3. THE LAW OF CONSERVATION OF ENERGY FOR AN INFINITESIMAL PARTICLE

When solving practical problems, including the formulation of boundary conditions, there is a need for an additional type of energy, such as the energy of external forces which is actually an energy equivalent of the effect of replaced external forces on the motion of the considered mechanical system. Particularly, for an infinitesimal particle within a body, the energy equivalent of the effect of external forces δE_e on the displacements of the particle boundaries can be determined by the scalar product of the forces acting on the boundaries $\delta\mathbf{P}$ and their displacement $\mathrm{d}\mathbf{r}$:

$$\mathrm{d}\delta E_\mathrm{e} = \sum(\delta\mathbf{P}\cdot\mathrm{d}\mathbf{r}) = \sum(\delta\mathbf{P}\cdot\mathbf{v})\mathrm{d}t. \qquad (11)$$

The summation on the right-hand side must be carried out over all surfaces bounding the given particle. We use the notion of the surface density of forces

$$\tau_{\alpha i} = \frac{\delta P_{\alpha i}}{\delta\beta\delta\gamma},\ \tau_{\beta i} = \frac{\delta P_{\beta i}}{\delta\alpha\delta\gamma},\ \tau_{\gamma i} = \frac{\delta P_{\gamma i}}{\delta\alpha\delta\beta}, \qquad (12)$$

where the subscript $p \in (\alpha, \beta, \gamma)$ for the forces δP_{pi} and stresses τ_{pi} indicates the normal direction to the considered surface area in the initial state, and the subscript $i \in (x, y, z)$ denotes the projection direction of the force and velocity. Taking into account that the forces and velocities can change on opposite faces, assuming all functions to be differentiable and given in the Lagrangian variables, and using the summation convention for repeated indices, we obtain the following expression for the rate of change in the specific energy of external forces [3, 4]:

$$\omega = \mathrm{d}\delta E_\mathrm{e}/(\delta V_0\mathrm{d}t) = \tau_{pi}x_{i,tp} + x_{i,tp}\partial\tau_{pi}/\partial\alpha_p. \qquad (13)$$

The stresses τ_{pi} form an asymmetric second rank tensor, which will be called the Lagrangian stresses for brevity. The stresses τ_{pi} are similar to the Piola–Kirchhoff stresses [1, 5], but they cannot be identified because they differ in the domain of arguments (Lagrangian variables) and the choice of the origin of the scale of mean stresses which can be shifted relative to the generally accepted one [6].

The energy conservation law for an infinitesimal particle is written as

$$\mathrm{d}\delta E = \sum_{i=1}^{13}\mathrm{d}\delta E_i(k_i\xi_i) - \mathrm{d}\delta E_\mathrm{e} = 0. \qquad (14)$$

Representing the increment of the energy of external forces (13) through the volume density of the energy change rate $\omega = \mathrm{d}\delta E_\mathrm{e}/(\mathrm{d}t\delta V_0)$, energy conservation law (14) can be written as the energy balance relation:

$$\mathrm{d}\delta E = \delta V_0(k_1\xi_{1,t} + k_2\xi_{2,t} + k_3\xi_{3,t} + \ldots + k_{13}\xi_{13,t} - \omega)\mathrm{d}t = 0. \qquad (15)$$

Excluding processes with dissipation, i.e., without taking into account the invariants associated with time integration, and using the generally accepted relationships for the potential and kinetic energy (the z axis is directed upward)

$$\mathrm{d}\delta E_1 = \rho_0 g z_t \delta V_0 \mathrm{d}t,$$

$$\mathrm{d}\delta E_2 = \rho_0(x_t x_{tt} + y_t y_{tt} + z_t z_{tt})\delta V_0\mathrm{d}t,$$

we have on the right-hand side of Eq. (15)

$$\mathrm{d}\delta E_1 + \mathrm{d}\delta E_2 + \mathrm{d}\delta E_5 + \mathrm{d}\delta E_6 + \mathrm{d}\delta E_7 = \omega\delta V_0\mathrm{d}t,$$

or

$$k_5\frac{\partial\xi_5}{\partial t} + k_6\frac{\partial\xi_6}{\partial t} + k_7\frac{\partial\xi_7}{\partial t} = \tau_{pi}x_{i,tp} + x_{i,t}\left(\frac{\partial\tau_{pi}}{\partial\alpha_p} + \rho_0(g_i - x_{i,tt})\right). \qquad (16)$$

Since the energy should be independent of subjective factors, including the choice of the reference velocity frame, the sum of the last three terms in Eq. (16) should vanish:

$$x_{i,t}\left(\frac{\partial\tau_{pi}}{\partial\alpha_p}+\rho_0 g_i-\rho_0 x_{i,tt}\right)=0.$$

Each of the brackets is usually set to zero:

$$\frac{\partial\tau_{pi}}{\partial\alpha_p}+\rho_0 g_i-\rho_0 x_{i,tt}=0, \quad (17)$$

but this assumption may lead to the loss of some possible solutions. Neglecting the gravitational forces, instead of Eq. (17) we obtain

$$\frac{\partial\tau_{pi}}{\partial\alpha_p}=\rho_0 x_{i,tt}. \quad (18)$$

In the absence of accelerations, the right-hand side vanishes:

$$\frac{\partial\tau_{pi}}{\partial\alpha_{\mathrm{p}}}=0. \quad (19)$$

Equations (18) and (19) are called the differential equations of motion and equilibrium [1, 5]. Taking into account the last equations, the expression for the power (13) takes the form

$$\omega=\frac{\mathrm{d}\delta E_{\mathrm{e}}}{\mathrm{d}t\delta V_0}=\tau_{pi}x_{i,tp}. \quad (20)$$

The total power

$$W=\int\limits_{V_0}\tau_{pi}x_{i,tp}\delta V_0=\int\limits_{V_0}\omega(x_{i,t\alpha},x_{i,t\beta},x_{i,t\gamma})\delta\alpha\delta\beta\delta\gamma \quad (21)$$

can be considered as a functional in which the integrand arguments are the velocities of the generalized coordinates $x_{i,tp}$. The extremum conditions of the functional [7]

$$\frac{\partial}{\partial\alpha}\left(\frac{\partial\omega}{\partial x_{i,t\alpha}}\right)+\frac{\partial}{\partial\beta}\left(\frac{\partial\omega}{\partial x_{i,t\beta}}\right)+\frac{\partial}{\partial\gamma}\left(\frac{\partial\omega}{\partial x_{i,t\gamma}}\right)=0,$$

with allowance for power (20), correspond to the equations $\partial\tau_{pi}/\partial\alpha_p=0$, that coincide with equilibrium equations (19). Hence it follows that equations of motion (1) and the corresponding velocity field with directional derivatives $x_{i,tp}$ must provide the minimum total power of external forces which is determined by the Lagrangian stresses τ_{pi} and is expended in the deformation of the considered body volume V_0 under nearly static conditions. Such a velocity field is conventionally called actual [2].

For each problem of deformation, other equations of motion and velocity fields can be proposed which differ from the actual ones, but are consistent with the conditions at the boundaries of the body. Such velocity fields are called kinematically possible. The power of external forces calculated on the basis of such velocity fields and the corresponding equations of motion must exceed its actual value. Of all the kinematically possible velocity fields, the fields corresponding to the minimum total power (21) expended in the deformation of the body will be the closest to the actual field.

4. LAGRANGIAN STRESSES AND STRAINS

The important characteristics of the deformation processes are Lagrangian stresses (12), which are local generalized forces and characterize energy fluxes at the boundaries of an infinitesimal particle via the surface force density or the rate of change in the volume density of the particle energy with the corresponding tensor components $x_{i,t\alpha_p}$. According to the differential equations of motion (17), the Lagrangian stresses determine the particle acceleration and system behavior. The Lagrangian stresses τ_{pi} must be dependent on the external forces and material properties which appear in the law of conservation of energy (16) as the coefficients k_5-k_{13}.

The dependence of the stresses on the material properties and strain state can be found from experimental studies, as well as from the general thermodynamic principles [8], to which the energy conservation law can be easily assigned. For the reversible deformation region (without account for the invariants associated with time integration), instead of Eq. (14), we have

$$\mathrm{d}\delta E_1+\mathrm{d}\delta E_2+\mathrm{d}\delta E_3+\mathrm{d}\delta E_5+\mathrm{d}\delta E_6+\mathrm{d}\delta E_7=\omega\delta V_0\mathrm{d}t.$$

Taking into account that the increment of the third invariant involves third order time derivatives of the current coordinates, which may not appear in the generalized equation of motion (16), and taking into account the differential equations of motion (17), we will use only the relation

$$k_5\frac{\partial\xi_5}{\partial t}+k_6\frac{\partial\xi_6}{\partial t}+k_7\frac{\partial\xi_7}{\partial t}=\tau_{pi}x_{i,tp}.$$

Invariants (4) are not included in the energy balance of reversible deformation because their time derivatives contain multipliers of the type $x_{i,ttp}$, that do not enter into Eq. (13) for the power of external forces.

Equating the coefficients at equal multipliers $x_{i,tp}$, we obtain relationships between the components of stresses τ_{pi}, elements of the tensor $\partial x_i/\partial\alpha_p\equiv x_{i,\alpha_p}$, and the coefficients k_i, characterizing the physical properties of the material:

$$\tau_{pi}=k_5\delta_{pi}+2k_6x_{i,p}+k_7\tilde{x}_{i,p}. \quad (22)$$

In fact, Eqs. (22) are analogs of Hooke's law for the region of elastic deformation of materials. In Eqs. (22) and further on, $\tilde{x}_{i,p}$ are the algebraic complements of the elements $x_{i,p}$ of the matrix $\partial x_i/\partial\alpha_p\equiv x_{i,\alpha_p}$, the tensor $\delta_{pi}=1$ for $\tau_{\alpha x},\tau_{\beta y},\tau_{\gamma z}$, and $\delta_{pi}=0$ for all other stresses not located on the main diagonal of the tensor τ_{pi}. In

the initial state when the matrix $\partial x_i/\partial\alpha_p \equiv x_{i,\alpha_p}$ is transformed into an identity matrix, the stress components not located on the main diagonal of the tensor τ_{pi}, become zero ($\tau_{\beta x} = \tau_{\gamma x} = \tau_{\alpha y} = \tau_{\gamma y} = \tau_{\alpha z} = \tau_{\beta z} = 0$). Only normal stresses on the faces of the original rectangular parallelepiped remain nonzero; they are all the same and depend on the coefficients k_5, k_6, k_7, at $t = 0$

$$\tau_{\alpha x} = \tau_{\beta y} = \tau_{\gamma z} = k_5 + 2k_6 + k_7. \quad (23)$$

Condition (23) does not exclude that these stresses in the initial state will be equal to zero if the coefficients $k_5 - k_7$ determine not only the properties, but also the choice of the scale of stresses τ_{pi}, by analogy with the potential energy of bodies in the gravitational field of the Earth.

Now let us find the relationship between the Lagrangian stresses τ_{pi} and Cauchy stresses σ_{ij} [1]. To do this, we consider energy balance (13) at the initial instant, when the Lagrangian stresses τ_{pi} coincide with the Cauchy stresses σ_{ij}

$$\frac{\mathrm{d}\delta E_\mathrm{e}}{\delta V \mathrm{d}t} = \sigma_{ij} x_{j,ti}. \quad (24)$$

Passing from the derivatives with respect to the Eulerian variables $\partial x_{i,t}/\partial x_j \equiv x_{i,tj}$ to the derivatives with respect to the Lagrangian variables $\partial x_{i,t}/\partial\alpha_\mathrm{p} \equiv x_{i,tp}$ with the use of general relations derived from equations of motion (1)

$$\frac{\partial f}{\partial x_i} = \frac{1}{R}\frac{\partial f}{\partial \alpha_p}\tilde{x}_{i,p},$$

and equating the coefficients at equal multipliers $x_{i,tp}$ on the right-hand sides of Eqs. (13) and (24), we obtain a system of linear equations

$$\tau_{pi} = \sigma_{ji}\tilde{x}_{j,p},$$

which formally coincide with the static conditions on the contour and determine the relationship between the Lagrangian and Cauchy stresses

$$\sigma_{ji} = \frac{\tau_{pi} x_{j,p}}{R}.$$

These equations are known as the relationships between the Cauchy and Piola–Kirchhoff stresses [4, 5]. However, as noted above, the Lagrangian stresses differ in the domain of variables and in the capability of changing the origin of the scale of mean stresses.

5. PHYSICAL PROPERTIES GOVERNING MOTION AND ENERGY

The scalar coefficients k_i, that allow the passage to unobservable energy parameters through observable and directly measurable kinematic parameters ξ_i should characterize the most important physical properties of the material or the medium in which motion takes place. Since these coefficients enter into the energy conservation law together with the invariants ξ_i, only 12 of the invariants can be independent. Likewise, the scalar coefficients k_i ($i = 1, ..., 12$) depend on the choice of the scale of the main type of energy, i.e., the choice of the value and dimension of one of them affects the values and dimensions of others. The relationship between these factors can be found by considering motion when the corresponding types of energy change.

For example, the relationship between the coefficients of displacements k_1 and velocities k_2, which characterize the potential and kinetic energy according to the generally accepted terminology, can be determined by considering the free fall of a body in the form of a material point in the absence of air resistance.

If the z axis is oriented toward the center of the Earth, the equations will read

$$x = \alpha,\ y = \beta,\ z = \gamma + u_z(\gamma, t).$$

The increments of potential energy $\mathrm{d}\delta E_1$ and kinetic energy $\mathrm{d}\delta E_2$ will be

$$\mathrm{d}\delta E_1 = -k_1 \delta V_0 z_t \mathrm{d}t < 0,$$

$$\mathrm{d}\delta E_2 = k_2 \delta V_0 \mathrm{d}(v_z^2) = 2k_2 \delta V_0 z_{tt} z_t \mathrm{d}t > 0.$$

From the law of conservation of energy

$$\mathrm{d}\delta E_1 + \mathrm{d}\delta E_2 = -k_1 z_t \delta V_0 \mathrm{d}t + 2k_2 z_t z_{tt} \delta V_0 \mathrm{d}t = 0$$

it follows that $k_1 = 2k_2 z_{tt}$. If we use the conventional notation for the acceleration of gravity $z_{tt} = g$, then $k_1 = 2k_2 g$. If not a particle but a body of mass m is considered, then $k_1 = mg$, and $k_2 = m/2$, which yields the conventional expressions for the kinetic and potential energy of the body

$$E_\mathrm{k} = mv^2/2,\ \Delta E_\mathrm{p} = mg\Delta z.$$

From the potential energy increment $\mathrm{d}E_\mathrm{p} = -mg\mathrm{d}z$ for the z axis directed toward the Earth's center or $\mathrm{d}E_\mathrm{p} = -mg\mathrm{d}z$ for the z axis directed from the Earth's center, the generalized force with the modulus $|Q_{pz}| = mg$ in the both cases is directed from the Earth's center and is opposite to the conventional direction of the body weight.

The coefficients $k_5 - k_7$ correspond to the properties of materials in the reversible deformation region. The coefficients of time-integral invariants, including those for path (5), should characterize dissipative processes, including processes associated with friction on the outer particle surfaces.

Taking into account Eq. (22), the Cauchy stresses σ_{ji}, can be represented, like the Lagrangian stresses τ_{pi}, in terms of the coefficients $k_5 - k_7$:

$$\sigma_{ji} = \frac{1}{R}(\tau_{\alpha i}x_{j,\alpha} + \tau_{\beta i}x_{j,\beta} + \tau_{\gamma i}x_{j,\gamma})$$
$$= \frac{1}{R}[k_5 x_{j,\alpha} + 2k_6(x_{i,\alpha}x_{j,\alpha} + x_{i,\beta}x_{j,\beta} + x_{i,\gamma}x_{j,\gamma}) + k_7(x_{j,\alpha}\tilde{x}_{i,\alpha} + x_{j,\beta}\tilde{x}_{i,\beta} + x_{j,\gamma}\tilde{x}_{i,\gamma})], \quad (25)$$

for example,

$$\sigma_{xx} = \frac{1}{R}(\tau_{\alpha x}x_\alpha + \tau_{\beta x}x_\beta + \tau_{\gamma x}x_\gamma) = \frac{1}{R}[k_5 x_\alpha + 2k_6(x_\alpha^2 + x_\beta^2 + x_\gamma^2) + k_7(x_\alpha\tilde{x}_\alpha + x_\beta\tilde{x}_\beta + x_\gamma\tilde{x}_\gamma)],$$
$$\sigma_{xy} = \frac{1}{R}(\tau_{\alpha y}x_\alpha + \tau_{\beta y}x_\beta + \tau_{\gamma y}x_\gamma) = \frac{1}{R}[k_5 x_\beta + 2k_6(x_\alpha y_\alpha + x_\beta y_\beta + x_\gamma y_\gamma) + k_7(x_\alpha\tilde{y}_\alpha + x_\beta\tilde{y}_\beta + x_\gamma\tilde{y}_\gamma)].$$

Taking into account that the multipliers of the coefficient k_7 in Eqs. (25) can take only two values:

$$x_{j,\alpha}\tilde{x}_{i,\alpha} + x_{j,\beta}\tilde{x}_{i,\beta} + x_{j,\gamma}\tilde{x}_{i,\gamma} = R \text{ at } i = j \text{ and}$$
$$x_{j,\alpha}\tilde{x}_{i,\alpha} + x_{j,\beta}\tilde{x}_{i,\beta} + x_{j,\gamma}\tilde{x}_{i,\gamma} = 0 \text{ at } i \neq j,$$

the coefficient k_7 actually enters only into the normal stresses

$$\sigma_{ii} = \frac{1}{R}(\tau_{\alpha i}x_{i,\alpha} + \tau_{\beta i}x_{i,\beta} + \tau_{\gamma i}x_{i,\gamma}) = \frac{1}{R}[k_5 x_{i,\alpha} + 2k_6(x_{i,\alpha}x_{i,\alpha} + x_{i,\beta}x_{i,\beta} + x_{i,\gamma}x_{i,\gamma}) + k_7 R].$$

The main invariant characteristic of the stress state may be the average Cauchy stress $\sigma = 1/3(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$, for which we obtain from Eq. (25):

$$3\sigma R = k_5\xi_5 + 2k_6\xi_6 + 3k_7\xi_7, \quad (26)$$

or, taking into account that $\xi_7 = R = \delta V / \delta V_0$:

$$\sigma = (k_5\xi_5 + 2k_6\xi_6)/(3R) + k_7.$$

The coefficient k_7 is an additive component of the mean stress. It can only affect the choice of the origin of the scale of mean stresses and cannot characterize its change that depends only on two coefficients k_5 and k_6:

$$\Delta\sigma = \Delta\frac{k_5\xi_5 + 2k_5\xi_6}{3R} = k_5\left(\frac{\xi_5}{3R} - 1\right) + 2k_6\left(\frac{\xi_6}{3R} - 1\right).$$

The mean stress in the initial state can be determined from Eq. (26) if we take into account the initial values of the invariants $R = 1, \xi_5 = 3, \xi_6 = 3, \xi_7 = 1$:

$$\sigma_0|_{t=0} = k_5 + 2k_6 + k_7,$$

which agrees with Eqs. (23). It can be assumed that the coefficient k_7 takes into account preliminary treatment, as well as plastic deformation, which can lead to hardening and a change in the energy state of the material.

We assume here that the mean stresses should not be regarded as zero in the initial state for different types of continua. Moreover, the value σ_0 can be considered as a physical property, namely, as the volumetric energy density associated with the three mentioned invariants in the initial state of the particle.

The necessity to account for the initial mean stress follows, e.g., from the comparison of the law of elastic volume change $\sigma = 3K\varepsilon$ with the Boyle–Mariotte law $pV = \text{const}$, which can be transformed to the form

$$p_1 - p_0 = p_1(V_1 - V_0)/V_0$$

and interpreted as a special case of the law of elastic volume change, if taking into account that $3\varepsilon \approx \Delta V/V_0$, and considering the bulk modulus K as the current pressure.

If the coefficients $k_5 - k_7$ are known, equations of motion (1) can be used to determine any kinematic and then energy and force functions, including the Lagrangian stresses via Eqs. (22) and Cauchy stresses via Eqs. (25).

Assuming that $k_5 = k_7 = 0$, we can greatly simplify the procedure of elastic deformation analysis, without interfering with classical mechanics. In particular, the terms of the Lagrangian stresses with the coefficients k_5 and k_7 have no effect on the differential equations of motion or equilibrium, because

$$\frac{\partial\tilde{x}_\alpha}{\partial\alpha} + \frac{\partial\tilde{x}_\beta}{\partial\beta} + \frac{\partial\tilde{x}_\gamma}{\partial\gamma} = 0,$$

and at $k_5 = \text{const}$ instead of Eqs. (18) and (19) we have

$$\frac{\partial^2 x_i}{\partial\alpha_p^2} = \frac{\rho_0}{2k_6}x_{i,tt} \text{ or } \frac{\partial^2 x_i}{\partial\alpha_p^2} = 0.$$

The equations of motion are reduced to three Poisson's or Laplace's equations, each of which contains only one unknown function. The methods of integrating such equations are well developed [9].

At $k_5 = 0$ or $\partial x_j/\partial\alpha_i = \partial x_i/\partial\alpha_j$ the tensor τ_{pi} becomes symmetric, like in classical mechanics. Of special importance is that in this case the Lagrangian stresses and strains differ only in the scale multiplier $\tau_{pi} = 2k_6 \times x_{i,p}$, the notion of stresses becomes redundant, and the material properties are determined only by one physical constant k_6. At $k_5 = k_7 = 0$, using the notation $2k_6 = \varphi$, instead of Eq. (26) we obtain

$$3\sigma R = \tau_{pi}x_{i,p} = \varphi\xi_6. \quad (27)$$

As was suggested in Ref. [5], the coefficient φ alone is sufficient to reliably describe deformation processes in the elastic region, and the transition from one scale of mean stresses to another can be performed by introducing the mean Lagrangian stress $\tau = \varphi e$ and transforming Eq. (27) to the form

$$3\sigma R = 3\tau e + \varphi\Gamma^2,$$

where $e = (e_\alpha + e_\beta + e_\gamma)/3$ is the mean change in the edge lengths of an infinitesimal parallelepiped $e_p^2 =$

$(\delta l/\delta l_0)_p^2 = x_p^2 + y_p^2 + z_p^2$, $p \in (\alpha, \beta, \gamma)$ [10]. Then, the relation $\sigma = \varphi e^2/R = \varphi/e = \varphi R^{-1/3}$ must hold true under multidirectional compression. For the transition to the ordinary scale, it is sufficient to assume $\tau = \varphi(e-1)$ and the mean Cauchy stress will determine the equation

$$\sigma = \tau e/R = \varphi(e-1)/e^2 = \varphi(1-R^{-1/3})/R^{1/3}.$$

For iron at $\varphi = 3K$ and K = 169 GPa, this relation differs from the experimentally determined one in the range up to 300 GPa [11]:

$$R = 1 + 5.286\times10^{-7}\sigma + 0.8\times10^{-12}\sigma^2$$

with an error of no more than 0.016%.

For linear stretching with the equations of motion

$$x = \alpha(1+\varepsilon_{xx}),\ y = \beta(1-\mu\varepsilon_{xx}),\ z = \gamma(1-\mu\varepsilon_{xx}),$$

where μ is the Poisson ratio, from Eq. (27) we have $\sigma_{xx} = \varphi\varepsilon_{xx}(1-2\mu)$ or $\sigma_{xx} = E\varepsilon_{xx}$, taking into account the relationship between the elastic moduli $E = 3K(1-2\mu)$.

The possibility of excluding the invariants ξ_5 and ξ_7 from the energy balance relation and the coefficients k_5, k_7 from Eqs. (22) for the deformation of solids can be justified by the fact that they actually account for the particle energy associated with the volume change and differ only by the constant multiplier. For the linear invariant ξ_5 for deformed solids, we can assume

$$\xi_5 = x_\alpha + y_\beta + z_\gamma = 3 + \varepsilon_x + \varepsilon_y + \varepsilon_z$$
$$= 3 + 3\varepsilon = 3(1+\varepsilon) \approx 3(1+\Delta V/V) \approx 3R,$$

which is 3 times as large as the invariant ξ_7, that determines the volume ratio R.

Thus, if the invariants ξ_5 and ξ_7 are retained in the energy balance relation, it is possible to reassess the energy accumulated by a particle of a solid body due to volume change. Probably, the influence of these invariants can be significant for materials, e.g., synthetic, with a wider range of volume changes.

The total energy expenditure in elastic deformation is taken into account by the quadratic invariant

$$\xi_6 = x_\alpha^2 + x_\beta^2 + x_\gamma^2 + y_\alpha^2 + \ldots z_\gamma^2 = 3e^2 + \Gamma^2.$$

The first term $3e^2$ on the right-hand side takes into account the volume change energy, and the second term Γ^2 accounts for the shape change energy. With the unchanged numerical value of the invariant ξ_6, the elastic volume change energy can be transformed into the elastic shape change energy and vice versa, which can be the basis of physical processes associated with self-oscillations and resonance.

Other possible interpretations of the coefficients k_5, k_6, k_7 were discussed in Ref. [3] with regard to conventional relations and knowledge about the physical properties of materials in the elastic and plastic deformation regions, including using the proportionality conditions for spherical tensors and deviatoric stresses and strains [6].

6. CONDITIONS FOR TRANSITION TO IRREVERSIBLE STRAINS

The energy-based model and the Lagrangian equations of motion describe the specific mechanisms of transition from reversible to irreversible strains. In this paper, the condition of plasticity implies the possibility of strain growth due to the energy accumulated in the deformed body. This state corresponds to the equation

$$k_5(\xi_5)_t + k_6(\xi_6)_t + k_7(\xi_7)_t = 0.$$

Further analysis will be carried out for the principal axes, assuming that at any time point we can choose an orthogonal coordinate system in which displacements are absent

$$k_5(x_{t\alpha} + y_{t\beta} + z_{t\gamma}) + 2k_6(x_\alpha x_{t\alpha} + y_\beta y_{t\beta} + z_\gamma z_{t\gamma}) + k_7(\xi_7)_t = 0. \quad (28)$$

For example, for multidirectional compression, instead of Eq. (28), we have

$$x_{t\alpha}(3k_5 + 6k_6 x_\alpha + 3k_7 x_\alpha^2) = 0,$$

and the possible solutions will be either $x_{t\alpha} = 0$, or $k_5 + 2k_6 x_\alpha + k_7 x_\alpha^2 = 0$.

From the energy point of view, the plasticity condition is reduced to the possibility of a consistent change of the three strain tensor invariants, due to which the left-hand side of Eq. (28) reduces to zero at any values of the coefficients $k_5 - k_7$. In this case, spontaneous motion is possible only by means of internal sources. The solution with any constant values of the coefficients $k_5 - k_7$ reduces to solving a system of three linear equations with respect to three unknown functions, namely, the derivatives of velocity components in the directions

$$x_{t\alpha} + y_{t\beta} + z_{t\gamma} = 0,\ x_\alpha x_{t\alpha} + y_\beta y_{t\beta} + z_\gamma z_{t\gamma} = 0,$$
$$\tilde{x}_\alpha x_{t\alpha} + \tilde{y}_\beta y_{t\beta} + \tilde{z}_\gamma z_{t\gamma} = 0. \quad (29)$$

Under hydrostatic loading, the system takes the form

$$3x_{t\alpha} = 0,\ 3x_\alpha x_{t\alpha} = 0,\ 3x_\alpha^2 x_{t\alpha} = 0$$

and has only a trivial solution $(x_{t\alpha} = 0)$, uniform tensile or compressive deformation cannot occur without changes in the accumulated energy. In other conditions, a nontrivial solution of the homogeneous system (29) for the derivatives $x_{t\alpha}, y_{t\beta}, z_{t\gamma}$ can exist if its main determinant is equal to zero

$$x_\alpha^2(z_\gamma - y_\beta) + y_\beta^2(x_\alpha - z_\gamma) + z_\gamma^2(y_\beta - x_\alpha) = 0. \quad (30)$$

In the case of linear stretching,

$$x = \alpha\exp(\varepsilon_x),\ y = \beta\exp(-\mu\varepsilon_x),\ z = \gamma\exp(-\mu\varepsilon_x),$$

Eq. (30) reduces to an identity due to $y_\beta = z_\gamma$, allowing the transition to irreversible deformation with preserving the direction of the principal axes and the homogeneous deformation state.

At the first stage $0 \le t \le t_s$, elastic deformation $0 \le \varepsilon_x \le \varepsilon_s$ with the transverse-to-longitudinal strain ratio $|\varepsilon_y/\varepsilon_x| = \mu$ causes energy accumulation, including accumulation due to volume change:

$$R = \exp[\varepsilon_x(1-2\mu)].$$

At the second stage $t_s \le t \le t_k$, a new component $\Delta\varepsilon_x \ge 0$ appears which differs from the strain of the first stage by the transverse-to-longitudinal strain ratio $|\Delta\varepsilon_y/\Delta\varepsilon_x| = \mu'$, and as a result the volume of particles returns to its initial value with simultaneous strain increase in the stretching direction:

$$x = \alpha\exp(\varepsilon_x + \Delta\varepsilon_x),\ y = \beta\exp(-\mu\varepsilon_x - \mu'\Delta\varepsilon_x),$$
$$z = \gamma\exp(-\mu\varepsilon_x - \mu'\Delta\varepsilon_x).$$

Beginning with the end of the first stage, with the ε_x value remaining unchanged and with increasing $\Delta\varepsilon_x$, the volume ratio of particles to whole specimen is

$$R = \exp[\varepsilon_s(1-2\mu) + \Delta\varepsilon_x(1-2\mu')].$$

Different μ' values will correspond to different degrees of volume recovery and different fractions of released energy. The volume of particles returns to its initial value if additional deformation satisfies the condition

$$\Delta\varepsilon_x = -\varepsilon_s(1-2\mu)/(1-2\mu').$$

For further extension of the specimen, the denominator should be negative, which is possible at $\mu' > 0.5$. Of course, the complete recovery of the volume is not necessary. The volume recovery also depends on the strain ratio $\Delta\varepsilon_x/\varepsilon_s$. The complete volume recovery is possible, e.g., if the strains in the first and second stages are equal, then $\mu' = 1-\mu$. In this case, at the beginning of the second stage at $\Delta\varepsilon_x = 0$ the main characteristics of the deformed state reach the values

$$R = \exp[\varepsilon_s(1-2\mu)],$$
$$3e = 3 + \varepsilon_s(1-2\mu) + 0.5\varepsilon_s^2(1+2\mu^2),$$
$$\Gamma_e^2 = 3 + 2\varepsilon_s(1-2\mu) + 2(1+2\mu)\varepsilon_s^2,$$
$$\Gamma^2 = 2/3[\varepsilon_s(1+\mu)]^2.$$

At the end of the second stage at $e_\alpha = x_\alpha = \exp(2\varepsilon_s)$ and $e_\beta = e_\gamma = \exp(-\varepsilon_s)$, they are $R = 1$, $e = 1+\varepsilon_s^2$, $\Gamma_e^2 = 3+12\varepsilon_s^2$, and $\Gamma^2 = 6\varepsilon_s^2$.

As a result, the volume, the average edge length, and the elastic energy return to their initial values with a significant change in the particle size, the standard deviation Γ^2 increases more than twice, and the power of external forces is almost completely dissipated into space.

The general equation for irreversible strains, without account for friction on the external surface of the body, takes the form

$$\Delta E = k_5\Delta\xi_5 + k_6\Delta\xi_6 + k_7\Delta\xi_7 + k_8\Delta\xi_8 + k_9\Delta\xi_9 + k_{10}\Delta\xi_{10} + k_{11}\Delta\xi_{11} + k_{12}\Delta\xi_{12} + k_{13}\Delta\xi_{13}.$$

The simplest example of a process with energy dissipation is the deformation during heating of an isotropic solid with the equations of motion

$$x_i = \alpha_i(1+\alpha_T\Delta T)$$

and the invariant increments

$$\Delta\xi_5 = 3\alpha_T\Delta T,\ \Delta\xi_6 = 6\alpha_T\Delta T + 3(\alpha_T\Delta T)^2,$$
$$\Delta\xi_7 = 3\alpha_T\Delta T + 3(\alpha_T\Delta T)^2 + (\alpha_T\Delta T)^3,$$
$$\Delta\xi_{11} = 3\alpha_T|\Delta T|,\ \Delta\xi_{12} = \sqrt{3}|\alpha_T\Delta T|,\ \Delta\xi_{13} = \alpha_T|\Delta T|,$$

where α_T is the linear expansion coefficient of the material in the considered temperature range $\Delta T = T - T_0$. The energy expended in deformation is

$$\Delta\delta E = \int_{T_0}^{T} c\delta m\,\mathrm{d}T = c_{\mathrm{av}}\delta m\Delta T \ \text{ or } \ \frac{\Delta\delta E}{\delta V_0} = c_{\mathrm{av}}\rho_0\Delta T, \quad (31)$$

where ρ_0 is the density of the material in the initial state, and c_{av} is the average heat capacity in the range from T_0 to T.

The properties associated with the coefficients $k_{11}-k_{13}$ are actually reduced to only σ_θ, because all three invariants are transformed to the form $\Delta\xi_i = A_i\alpha_T|\Delta T|$, where A_i are constant coefficients. The energy balance condition takes the form

$$\Delta E = 3\alpha_T\Delta T(k_5 + 2k_6 + k_7) + (3k_{11} + \sqrt{3}k_{12} + k_{13})\alpha_T|\Delta T| = 3\alpha_T\Delta T\sigma_0 + \alpha_T|\Delta T|\sigma_\theta, \quad (32)$$

where $\sigma_\theta = 3k_{11} + \sqrt{3}k_{12} + k_{13}$. Equating the right-hand sides of Eqs. (31) and (32), at $\sigma_0 = 0$ we have

$$\sigma_\theta = c_{\mathrm{av}}\rho_0/\alpha_T. \quad (33)$$

In accordance with the Grüneisen law [12], the right-hand side must remain constant, which confirms the previous suggestion that the quantities $k_{11}-k_{13}$ can be regarded as the physical properties of the medium associated with deformation processes, and, as follows from Ref. [11], the property σ_θ is more stable than each of the thermophysical characteristics entering into Eq. (33).

7. GENERALIZED FORCES IN ABSOLUTELY RIGID BODIES

For absolutely rigid bodies, equations of motion (1) can be derived in a general form. Let us consider, e.g., two-dimensional motion in the *xOy* plane of a body with two isolated fixed points *P* and *M* with the coordinates $P_0(\alpha_P, \beta_P)$ and $M_0(\alpha, \beta)$ at the initial instant. At an ar-

bitrary instant, the points occupy new positions with the coordinates $P(x_P, x_P)$ and $M(x, y)$. The angle between the straight line PM of length L_0 connecting these points and the x axis in the initial state will be denoted by φ_0, or by φ in the current state. The coordinates at the initial and current instants differ by the magnitude of the projections of the distance between P and M onto the coordinate axes

$$\alpha = \alpha_P + L_0 \cos\varphi_0, \ \beta = \beta_P + L_0 \sin\varphi_0, \quad (34)$$

$$x = x_P + L_0 \cos\varphi, \ y = y_P + L_0 \sin\varphi. \quad (35)$$

Passing to the angle increments $\Delta\varphi = \varphi - \varphi_0$, we obtain the equations of motion in the Lagrangian form

$$\begin{aligned} x &= x_P + (\alpha - \alpha_P)\cos\Delta\varphi - (\beta - \beta_P)\sin\Delta\varphi, \\ y &= y_P + (\alpha - \alpha_P)\sin\Delta\varphi + (\beta - \beta_P)\cos\Delta\varphi. \end{aligned} \quad (36)$$

Differentiation Eqs. (36) with respect to time, we derive relationships between the velocities of the points P and M with regard to the change of the angle φ in the space of Lagrangian and Eulerian variables

$$\begin{aligned} x_t &= (x_t)_P - \varphi_t[(\alpha - \alpha_P)\sin\Delta\varphi + (\beta - \beta_P)\cos\Delta\varphi], \\ y_t &= (y_t)_P + \varphi_t[(\alpha - \alpha_P)\cos\Delta\varphi - (\beta - \beta_P)\sin\Delta\varphi], \end{aligned} \quad (37)$$

$$\begin{aligned} x_t &= (x_t)_P - \varphi_t(y - y_P), \\ y_t &= (y_t)_P + \varphi_t(x - x_P), \end{aligned} \quad (38)$$

and accelerations

$$\begin{aligned} x_{tt} &= (x_{tt})_P - \varphi_{tt}[(\alpha - \alpha_P)\sin\Delta\varphi + (\beta - \beta_P) \\ &\times \cos\Delta\varphi] - \varphi_t^2[(\alpha - \alpha_P)\cos\Delta\varphi - (\beta - \beta_P)\sin\Delta\varphi], \\ y_{tt} &= (y_{tt})_P + \varphi_{tt}[(\alpha - \alpha_P)\cos\Delta\varphi - \beta - \beta_P) \\ &\times \sin\Delta\varphi] - \varphi_t^2[(\alpha - \alpha_P)\sin\Delta\varphi + (\beta - \beta_P)\cos\Delta\varphi], \end{aligned} \quad (39)$$

$$\begin{aligned} x_{tt} &= (x_{tt})_P - \varphi_{tt}(y - y_P) - \varphi_t^2(x - x_P), \\ y_{tt} &= (y_{tt})_P + \varphi_{tt}(x - x_P) - \varphi_t^2(y - y_P), \end{aligned} \quad (40)$$

where $\partial\varphi/\partial t \equiv \varphi_t$ and $\partial^2\varphi/\partial t^2 \equiv \varphi_{tt}$ are the angular velocity and the angular acceleration of the considered solid.

General equations (36)–(39) yield equations, as special cases, for translational motion ($\Delta\varphi = 0$) and rotation with respect to the fixed pole $P(a, b)$, when $\alpha_P = x_P = a = \text{const}$, $\beta_P = y_P = b = \text{const}$.

Using circular substitution, we can obtain the corresponding relationships for other planes, and the superposition principle [13] allows us to describe any kind of three-dimensional motion and use it for dynamic analysis.

Of special interest for absolutely rigid bodies is the integral energy of the whole body:

$$dE_i = Q_{ij} dq_j. \quad (41)$$

As in Eq. (9), the number of terms on the right-hand side of Eq. (41) depends not only on the motion pattern of the body, but also on the mechanism of its interaction with other bodies in the considered system. The integral forces Q_{ij} should be understood to mean the rate of change of energy E_i with the change in the corresponding kinematic coordinate q_j:

$$Q_{ij} = \frac{\partial E_i}{\partial q_j}.$$

Both local and total generalized forces are mathematical functions with possible mechanical interpretations that can be used to calculate the corresponding energy components.

The generalized kinematic coordinates q_j on the right-hand side of Eq. (41) need not necessarily be independent. For absolutely rigid bodies, the increment in the power of external forces can be determined through a set of forces applied to different points of the body, and the number of the kinematic coordinates q_j used in calculations can exceed the number of degrees of freedom of the given body. For example, for a body with only one degree of freedom oscillating about a fixed axis, the kinematic coordinates can be only either the angle of the body rotation, or the angle of rotation of coordinates of an arbitrary pole, e.g., the center of mass. In a more general case, two poles can be chosen, e.g., in dividing energy fluxes in the points where the power is removed [14]. Equation (41) should be regarded as an ordinary algebraic equation for the energy increment determination using sets of generalized forces convenient for the given problem.

Let us consider methods for calculating the kinetic energy of a body of volume V, density ρ, and mass m, which is determined by the second invariant from Eq. (2), for the case of two-dimensional motion:

$$E_k = 0.5\int_V \rho v^2 \delta V = 0.5\int_m v^2 \delta m = 0.5\int_m (x_t^2 + y_t^2)\delta m.$$

Taking into account relationships (38) between the velocity components of an arbitrary particle and the pole P, we find

$$\begin{aligned} E_k = 0.5\int_m \{&[(x_t)_P^2 + (y_t)_P^2 - 2\varphi_t[(x_t)_P(y - y_P) \\ &- (y_t)_P(x - x_P)] + \varphi_t^2[(x - x_P)^2 + (y - y_P)^2]\}\delta m. \end{aligned}$$

Using the notions of the center of mass $C(x_C, y_C)$ and the axial moment of inertia:

$$x_C m = \int_m x\delta m, \ y_C m = \int_m y\delta m,$$

$$J_P = \int_m [(x - x_P)^2 + (y - y_P)^2]\delta m,$$

we obtain the most general formula for the kinetic energy for the two-dimensional motion of an absolutely rigid body:

$$E_k = \frac{1}{2}mv_P^2 + \frac{1}{2}\varphi_t^2 J_P - \varphi_t m[(x_t)_P(y_C - y_P) - (y_t)_P(x_C - x_P)]. \quad (42)$$

The kinetic energy increment can be written as

$$\mathrm{d}E_k = m\{[(x_t)_P - \varphi_t(y_C - y_P)](x_{tt})_P + [(y_t)_P + \varphi_t(x_C - x_P)](y_{tt})_P\}\mathrm{d}t + \varphi_t\varphi_{tt}J_P\mathrm{d}t.$$

The generalized forces can be calculated by choosing increments of either linear and angular velocities $\mathrm{d}(x_t)_P$, $\mathrm{d}(y_t)_P$, $\mathrm{d}\varphi_t$, or displacements $\mathrm{d}x_P$, $\mathrm{d}y_P$, $\mathrm{d}\varphi$. Additionally, the pole may coincide with the center of mass or with the rotation axis (instantaneous center). Thus, we have six possible sets of generalized forces, each including forces and moments of forces. Their values and dimensions are different, but there is only one common feature: the energy increment value or the rate of energy change is the same if these forces are used correctly. The most ordinary are the generalized forces for the increments of the linear coordinates of the center of mass. They correspond to the simpler form of Eq. (42) at $(x_i)_P = (x_i)_C$:

$$E_k = 0.5mv_C^2 + 0.5\varphi_t^2 J_C. \quad (43)$$

The two terms on the right are usually associated with the translational and rotational motion of the body. However, the ratio between these parts depends on the choice of the pole. The only invariant is their sum that coincides with the rotational motion energy, if the pole coincides with the instantaneous center. Writing the increment of Eq. (43) in the form

$$\mathrm{d}E_k = m(x_{tt})_C\mathrm{d}x_C + m(y_{tt})_C\mathrm{d}y_C + J_C\varphi_{tt}\mathrm{d}\varphi,$$

we obtain the generalized forces

$$(F_x)_C = m(x_{tt})_C, (F_y)_C = m(y_{tt})_C, \quad M_C = \varphi_{tt}J_C, \quad (44)$$

the first two of which coincide in form with Newton's law (**F** = *m***a**) and may therefore be called the "Newtonian" forces [15].

In the case when the pole coincides with the instantaneous center, the generalized forces are the centrifugal forces [16] characterizing the rate of change in the kinetic energy of absolutely rigid bodies with the rates of change in the distance between the center of mass and the instantaneous center. Since the position of these points does not depend on the choice of the observer coordinate system, the centrifugal forces are less dependent on subjective factors.

Equation (43) is written in invariant form and is applicable not only to the two-dimensional, but also to the three-dimensional motion of an absolutely rigid body.

The potential energy of a body of mass m (the z axis directed from the Earth's center) can be changed by the increment of the coordinate in the corresponding direction (or its rate of change for the power of potential forces W_p):

$$\mathrm{d}E_\mathrm{p} = mg\mathrm{d}z, \; W_\mathrm{p} = \frac{\mathrm{d}E_\mathrm{p}}{\mathrm{d}t} = mgz_t. \quad (45)$$

Based on form (45), we choose the method for determining the energy of external forces (11) on the surface of infinitesimal particles. However, volume-integral generalized forces for solids should take into account the power increment not only for linear, but also for angular displacements:

$$\mathrm{d}E_e = Q_i\mathrm{d}x_i + M_i\mathrm{d}\varphi_i.$$

Theoretical mechanics usually employs the term "point of force application". Taking into account the energy-related meaning of the generalized forces and the possibility of an arbitrary choice of the pole, it is more correct to call them the reference points of generalized forces.

The main difference of volume-integral generalized forces for absolutely rigid bodies is their possible diversity, including the appearance of passive forces orthogonal to their reference point velocity, and therefore their power is equal to zero. These forces include in particular the centrifugal forces arising when the body rotates relative to a fixed axis. The description of the power of external forces implies much more cases when passive forces may arise.

The passive forces should be referred to potential forces. They often ensure the implementation of kinematic relationships and can produce power when these relationships change. The passive forces can generate other forces that have a different direction, e.g., frictional forces on contact surfaces. Forces on fixed supports of mechanisms must also be considered as passive. In accordance with the accepted practice, the passive forces are calculated using statics equations, but the obtained results may be underestimated compared to if they are calculated from an analysis of energy fluxes.

The statement about the ambiguity of the passive forces arising in both movable and fixed supports is one of the important results of transition to the energy-based interpretation of generalized forces and methods for their determination. The maximum possible values of passive forces must be taken into account in the design of mechanisms, as well as in testing their strength, stability, and reliability. The transition to scalar invariants is actually the only way to unambiguously describe the state of system elements.

The method of dynamic analysis of various mechanisms developed using the energy-based model allows

the energy conservation law to be satisfied for any part of the studied system and within an arbitrary time interval [14].

8. CONCLUSIONS

A concept was proposed in which energy as a generalized scalar function of any kind of motion is presented as a sum of terms, each of which depends on only one invariant of the Lagrangian equations of motion. The use of the energy conservation law makes it possible to derive basic equations of classical mechanics, including differential equations of motion and equilibrium, as well as stress-strain relationships transformed to the space of Lagrangian variables. The proposed concept opens up new possibilities for solving various problems, particularly the problems of validating the physical properties of materials included in the mathematical formulation of energy and methods for their determination, as well as the possibility of changing the scale of mean stresses with account for the volumetric energy density of particles in their initial state. We discussed the aspects concerning the multiple choice of generalized forces for absolutely rigid bodies, including passive forces in contacts between adjacent bodies of a mechanical system.

Within the energy-based model of mechanics, it was proposed to consider Newton's second law as a means of determining the generalized forces characterizing the kinetic energy increment of an absolutely rigid body with increments in the distance between the origin of the observer coordinate system and the center of mass of the moving body.

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